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# Mesoscopic modelling of financial markets

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## Abstract

We derive a mesoscopic description of the behavior of a simple financial market where the agents can create their own portfolio between two investments alternatives: a stock and a bond. The model is derived starting from the Levy-Levy-Solomon microscopic model [13, 14] using the methods of kinetic theory and consists of a linear Boltzmann equation for the wealth distribution of the agents coupled with an equation for the price of the stock. From this model under a suitable scaling we derive a Fokker-Planck equation and show that the equation admits a self-similar lognormal behavior. Several numerical examples are also reported to validate our analysis.

**Keywords:** wealth distribution, power-law tails, stock market, self-similarity, kinetic equations.

## 1 Introduction

In recent years there has been a constantly growing interest of physicists in new interdisciplinary areas such as sociology and economy originating what is today named socio-economical physics[1, 3, 4, 6, 9, 12, 13, 19, 23]. This new area in physics borrowed several methods and tools from classical statistical mechanics, where complex behavior arises from relatively simple rules due to the interaction of a large number of components. The motivation behind this is the attempt to identify and characterize universal and non-universal features in economical data in general.

The large part of the research in this area is concerned with power-law tails with universal exponents as was predicted more than one century ago by Pareto[4, 15, 22]. In particular, by identifying the wealth in an economic system with the energy of the physical system, the application of statistical physics makes it possible to understand better the development of tails in wealth distributions. Starting from the microscopic dynamics, mesoscopic models can be derived with the tools of classical kinetic theory of fluids[1, 5, 6, 7, 12, 20, 21, 24].

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At variance with microscopic dynamics where their behavior often can be studied only empirically through computer simulations, kinetic models based on PDEs allow to derive analytically general information on the model and its asymptotic behavior. For example, the knowledge of the large-wealth behavior is of primary importance, since it determines a posteriori if the model can fit data of real economies.

In some recent papers the explicit emergence of power laws in the wealth distribution, with Pareto index strictly larger the one, has been proved for open market economies where agents can interact through binary exchanges together with a simple source of speculative trading [1, 5, 21, 24].

The present work is motivated by the necessity to have a more realistic description of the speculative dynamics in the above models. To this goal we derive a mesoscopic description of the behavior of a simple financial market where a population of homogeneous agents can create their own portfolio between two investments alternatives: a stock and a bond. The model is closely related to the Levy-Levy-Solomon (LLS) microscopic model in finance [13, 14]. This model attempted to construct from simple rules complex behavior that could then mimic the market and explain the price formation mechanism. As a first step towards a more realistic description, we derive and analyze the model in the case of a single stock and under the assumption that the optimal proportion of investments is a function of the price only. In principle several generalizations are possible (different stocks, heterogeneous agents, a time dependent optimal proportion of investments, ...), and we leave them for future investigations.

In our non-stationary financial market model the average wealth is not conserved and produces the price variations. Let us point out that, even if the model is linear since no binary interaction dynamic between agents is present, the study of the large time behavior is not immediate. In fact, despite conservation of the total number of agents we don't have any other additional conservation equation or entropy dissipation. Although we prove that the moments do not grow more then exponentially the determination of an explicit form of the asymptotic wealth distribution of the kinetic equation remains difficult and requires the use of suitable numerical methods.

A complementary method to extract information on the tails is linked to the possibility to obtain particular asymptotics which maintain the characteristics of the solution to the original problem for large times. Following the analysis in [5] we shall prove that the Boltzmann model converges in a suitable asymptotic limit towards a convection-diffusion equation of Fokker-Planck type for the distribution of wealth among individuals. Other Fokker-Planck equations were obtained using different approaches in [1, 26, 17].

In this case, however, we can show that the Fokker-Planck equation admits self-similar solutions that can be computed explicitly and which are lognormal distributions. One is then lead to the conclusion that the formation of Pareto tails in the wealth distribution observed in [1, 5] is a consequence of the interplay between the conservative binary exchanges having the effect of redistribute wealth among agents and the speculative trading causing the grow of the mean wealth and social inequalities.

The rest of the paper is organized as follows. In the next section, we introduce briefly the microscopic dynamic of the LLS model. The mesoscopic model is then derived in Section 3 and its properties discussed in Section 4. These properties justify the asymptotic procedures performed in section 5. The model behavior together with its asymptotic limit

is illustrated by several numerical results in section 6. Some conclusions and remarks on future developments are then made in the last section.

## 2 The microscopic dynamic

Let us consider a set of financial agents  $i = 1, \dots, N$  who can create their own portfolio between two alternative investments: a stock and a bond. We denote by  $w_i$  the wealth of agent  $i$  and by  $n_i$  the number of stocks of the agent. Additionally we use the notations  $S$  for the price of the stock and  $n$  for the total number of stocks.

The essence of the dynamic is the choice of the agent's portfolio. More precisely at each time step each agent selects which fraction of wealth to invest in bonds and which fraction in stocks. We indicate with  $r$  the (constant) interest rate of bonds.

To simplify the notations let us neglect for the moment the effects due to the stochastic nature of the process, the presence of dividends, and so on. Thus if an agent has invested  $\gamma_i w_i$  of its wealth in stocks and  $(1 - \gamma_i)w_i$  of its wealth in bonds at the next time step in the dynamic he will achieve the new wealth value

$$w'_i = (1 - \gamma_i)w_i(1 + r) + \gamma_i w_i(1 + x'), \quad (1)$$

where  $x'$  is the rate of return of the stock given by

$$x' = \frac{S' - S}{S}, \quad (2)$$

and  $S'$  is the new price of the stock.

Since we have the identity

$$\gamma_i w_i = n_i S, \quad (3)$$

we can also write

$$w'_i = w_i + w_i(1 - \gamma_i)r + w_i \gamma_i \left( \frac{S' - S}{S} \right) \quad (4)$$

$$= w_i + (w_i - n_i S)r + n_i(S' - S). \quad (5)$$

Note that independently of the number of stocks of the agent at the next time level, it is only the price variation of the stock (which is unknown) that characterizes the gain or loss of the agent on the stock market at this stage.

The dynamic now is based on the agent choice of the new fraction of wealth he wants to invest in stocks at the next stage. Different models can be used for this (see [14]). For example maximizing a von Neumann-Morgenstern utility function with a constant risk aversion of the type

$$U(w) = \frac{w^{1-\alpha}}{1-\alpha}, \quad (6)$$

where  $\alpha$  is the risk aversion parameter. As they don't know the future stock price  $S'$  the investors estimate the stock's next period return distribution and find an optimal mix of the stock and the bond that maximizes their expected utility  $E[U]$ . In practice, for any hypothetical price  $S^h$ , each investor find the optimal proportion  $\gamma_i^h$  which maximizes his/her

expected utility. Note that, if we assume that all investors have the same risk aversion  $\alpha$  then they will have the same proportion of investment in stocks regardless of their wealth, thus  $\gamma_i^h(S^h) = \gamma^h(S^h)$ .

Following [14], each agent formulates a demand curve

$$n_i^h = n_i^h(S^h) = \frac{\gamma^h(S^h)w_i^h(S^h)}{S^h}$$

characterizing the desired number of stocks as a function of the hypothetical stock price  $S^h$  and the hypothetical new wealth value  $w_i^h$  obtained from (1) with  $S' = S^h$ . This number of share demands is a monotonically decreasing function of the hypothetical price  $S^h$ . As the total number of stocks

$$n = \sum_{i=1}^N n_i, \quad (7)$$

is preserved the new price of the stock at the next time level is given by the market clearance condition. Thus the new stock price  $S'$  is the unique price at which the total demands equals the supply

$$\sum_{i=1}^N n_i^h(S') = n. \quad (8)$$

This will fix the value  $w'$  in (1) and the model can be advanced to the next time level. To make the model more realistic typically a source of stochastic noise, which characterizes all factors causing the investor to deviate from his optimal portfolio, is introduced in the proportion of investments  $\gamma_i$  and in the rate of return of the stock  $x'$ .

### 3 Kinetic modelling

We define  $f = f(w, t)$ ,  $w \in \mathbb{R}_+$ ,  $t > 0$  the distribution of wealth  $w$ , which represents the probability for an agent to have a wealth  $w$ . We assume that at time  $t$  the percentage of wealth invested is of the form  $\gamma(\xi) = \mu(S) + \xi$ , where  $\xi$  is a random variable in  $[-z, z]$ ,  $z = \min\{-\mu(S), 1 - \mu(S)\}$  is distributed accordingly to some probability density  $\Phi(\mu(S), \xi)$  with zero mean and variance  $\zeta^2$ . This probability density characterizes the individual strategy of an agent around the optimal choice  $\mu(S)$ . We assumed  $\Phi$  to be independent of the wealth of the agent. Here the optimal demand curve  $\mu(\cdot)$  is assumed to be a given monotonically non increasing function of the price  $S \geq 0$  such that  $0 < \mu(0) < 1$ .

Note that given  $f(w, t)$  the actual stock price  $S$  satisfy the demand-supply relation

$$S = \frac{1}{n} E[\gamma w], \quad (9)$$

where  $E[X]$  denotes the mathematical expectation of the random variable  $X$  and  $f(w, t)$  has been normalized

$$\int_0^\infty f(w, t) dw = 1.$$

More precisely, since  $\gamma$  and  $w$  are independent, at each time  $t$  the price  $S(t)$  satisfies

$$S(t) = \frac{1}{n} E[\gamma] E[w] = \frac{1}{n} \mu(S(t)) \bar{w}(t), \quad (10)$$

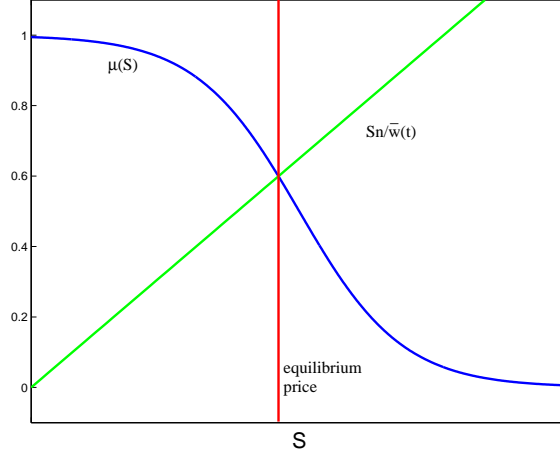


Figure 1: Example of equilibrium price

with

$$\bar{w}(t) \stackrel{def}{=} E[w] = \int_0^\infty f(w, t) w dw, \quad (11)$$

being the mean wealth and by construction

$$\mu(S) = \int \Phi(\mu(S), \xi) \xi d\xi.$$

At the next round in the market the new wealth of the investor will depend on the future price  $S'$  and the percentage of wealth invested  $\gamma$  accordingly with

$$w'(S', \gamma, \eta) = (1 - \gamma)w(1 + r) + \gamma w(1 + x(S', \eta)), \quad (12)$$

where the expected rate of return of stocks is given by

$$x(S', \eta) = \frac{S' - S + D + \eta}{S}. \quad (13)$$

In the above relation  $D \geq 0$  represent a constant dividend payed by the company and  $\eta$  is a random variable distributed accordingly to  $\Theta(\eta)$  with zero mean and variance  $\sigma^2$  which takes into account fluctuations due to price uncertainty and dividends[14, 11]. We assume  $\eta$  to take values in  $[-d, d]$  with  $0 < d \leq S' + D$  so that  $w' \geq 0$  and thus negative wealths are not allowed in the model. Note that equation (13) requires to estimate the future price  $S'$  which is unknown.

The dynamic is then determined by the agent's new fraction of wealth invested in stocks  $\gamma'(\xi') = \mu(S') + \xi'$  where  $\xi'$  is a random variable in  $[-z', z']$ , where  $z' = \min\{\mu(S'), 1 - \mu(S')\}$  distributed accordingly to  $\Phi(\mu(S'), \xi')$ . We have the demand-supply relation

$$S' = \frac{1}{n} E[\gamma' w'], \quad (14)$$

which permits to write the following equation for the future price

$$S' = \frac{1}{n} E[\gamma'] E[w'] = \frac{1}{n} \mu(S') E[w']. \quad (15)$$

Now

$$w'(S', \gamma, \eta) = w(1 + r) + \gamma w(x(S', \eta) - r), \quad (16)$$

thus

$$E[w'] = E[w](1 + r) + E[\gamma w](E[x(S', \eta)] - r), \quad (17)$$

$$= \bar{w}(t)(1 + r) + \mu(S) \bar{w}(t) \left( \frac{S' - S + D}{S} - r \right). \quad (18)$$

This gives the identity

$$S' = \frac{1}{n} \mu(S') \bar{w}(t) \left[ (1 + r) + \mu(S) \left( \frac{S' - S + D}{S} - r \right) \right]. \quad (19)$$

Using equation (10) we can eliminate the dependence on the mean wealth and write

$$\begin{aligned} S' &= \frac{\mu(S')}{\mu(S)} [(1 - \mu(S))S(1 + r) + \mu(S)(S' + D)] \\ &= \frac{(1 - \mu(S))\mu(S')}{(1 - \mu(S'))\mu(S)} (1 + r)S + \frac{\mu(S')}{1 - \mu(S')} D. \end{aligned} \quad (20)$$

### Remark 3.1

The equation for the future price deserves some remarks.

- Equation (20) determines implicitly the future value of the stock price. Let us set

$$g(S) = \frac{1 - \mu(S)}{\mu(S)} S.$$

Then the future price is given by the equation

$$g(S') = g(S)(1 + r) + D$$

for a given  $S$ . Note that

$$\frac{dg(S)}{dS} = -\frac{d\mu(S)}{dS} \frac{S}{\mu(S)^2} + \frac{1 - \mu(S)}{\mu(S)} > 0,$$

then the function  $g(S)$  is monotonically increasing with respect to  $S$ . This guarantees the existence of a unique solution

$$S' = g^{-1}(g(S)(1 + r) + D) > S. \quad (21)$$

Moreover if  $r = 0$  and  $D = 0$  the unique solution is  $S' = S$  and the price remains unchanged in time.

For the average stocks return we have

$$\bar{x}(S') - r = \frac{(\mu(S') - \mu(S))(1 + r)}{(1 - \mu(S'))\mu(S)} + \frac{\mu(S')D}{S(1 - \mu(S'))}, \quad (22)$$

where

$$\bar{x}(S') = E[x(S', \eta)] = \frac{S' - S + D}{S}. \quad (23)$$

Now the right hand side of (22) has non constant sign since  $\mu(S') \leq \mu(S)$ . In particular the average stocks return is above the bonds rate  $r$  only if the (negative) rate of variation of the investments is above a certain threshold

$$\frac{\mu(S') - \mu(S)}{\mu(S)\mu(S')} S \geq -\frac{D}{(1 + r)}.$$

- In the constant investment case  $\mu(\cdot) = C$ , with  $C \in (0, 1)$  constant then we have  $g(S) = (1 - C)S/C$  and

$$S' = (1 + r)S + \frac{C}{1 - C}D,$$

which corresponds to a dynamic of grow of the prices at rate  $r$ . As a consequence the average stocks return is always larger then the constant return of the bonds

$$\bar{x}(S') - r = \frac{D}{S(1 - C)} \geq 0.$$

By standard methods of kinetic theory [2] the microscopic dynamics of agents originate the following linear kinetic equation for the evolution of the wealth distribution

$$\frac{\partial f(w, t)}{\partial t} = \int_{-d}^d \int_{-z}^z \left( \beta('w \rightarrow w) \frac{1}{j(\xi, \eta, t)} f('w, t) - \beta(w \rightarrow w') f(w, t) \right) d\xi d\eta. \quad (24)$$

The above equations takes into account all possible variations that can occur to the distribution of a given wealth  $w$ . The first part of the integral on the right hand side takes into account all possible gains of the test wealth  $w$  coming from a pre-trading wealth  $'w$ . The function  $\beta('w \rightarrow w)$  gives the probability per unit time of this process.

Thus  $'w$  is obtained simply by inverting the dynamics to get

$$'w = \frac{w}{j(\xi, \eta, t)}, \quad j(\xi, \eta, t) = 1 + r + \gamma(\xi)(x(S', \eta) - r), \quad (25)$$

where the value  $S'$  is given as the unique fixed point of (15).

The presence of the term  $j$  in the integral is needed in order to preserve the total number of agents

$$\frac{d}{dt} \int_0^\infty f(w, t) dw = 0.$$

The second part of the integral on the right hand side of (24) is a negative term that takes into account all possible losses of wealth  $w$  as a consequence of the direct dynamic (12). The rate of this process now being  $\beta(w \rightarrow w')$ . In our case the kernel  $\beta$  takes the form

$$\beta(w \rightarrow w') = \Phi(\mu(S), \xi) \Theta(\eta). \quad (26)$$



The distribution function  $\Phi(\mu(S), \xi)$  together with the function  $\mu(\cdot)$  characterize the behavior of the agents on the market (more precisely they characterize the way the agents invest their wealth as a function of the actual price of the stock).

**Remark 3.2** In the derivation of the kinetic equation we assumed for simplicity that the actual demand curve  $\mu(\cdot)$  which gives the optimal proportion of investments is a function of the price only. In reality the demand curve should change at each market iteration and so should depend also on time. In the general case where each agent has a wealth dependent individual strategy one should consider the distribution  $f(\gamma, w, t)$  of agents having a fraction  $\gamma$  of their wealth  $w$  invested in stocks.

## 4 Properties of the kinetic equation

We will start our analysis by introducing some notations. Let  $\mathcal{M}_0$  the space of all probability measures in  $\mathbb{R}_+$  and by

$$\mathcal{M}_p = \left\{ \Psi \in \mathcal{M}_0 : \int_{\mathbb{R}_+} |\vartheta|^p \Psi(\vartheta) d\vartheta < +\infty, p \geq 0 \right\}, \quad (27)$$

the space of all Borel probability measures of finite momentum of order  $p$ , equipped with the topology of the weak convergence of the measures.

Let  $\mathcal{F}_p(\mathbb{R}_+)$ ,  $p > 1$  be the class of all real functions on  $\mathbb{R}_+$  such that  $g(0) = g'(0) = 0$ , and  $g^{(m)}(v)$  is Hölder continuous of order  $\delta$ ,

$$\|g^{(m)}\|_\delta = \sup_{v \neq w} \frac{|g^{(m)}(v) - g^{(m)}(w)|}{|v - w|^\delta} < \infty, \quad (28)$$

the integer  $m$  and the number  $0 < \delta \leq 1$  are such that  $m + \delta = p$ , and  $g^{(m)}$  denotes the  $m$ -th derivative of  $g$ .

Clearly the symmetric probability density  $\Theta$  which characterizes the stock returns belongs to  $\mathcal{M}_p$ , for all  $p > 0$  since

$$\int_{-d}^d |\eta|^p \Theta(\eta) d\eta \leq |d|^p.$$

Moreover, to simplify computations, we assume that this density is obtained from a given random variable  $Y$  with zero mean and unit variance. Thus  $\Theta$  of variance  $\sigma^2$  is the density of  $\sigma Y$ . By this assumption, we can easily obtain the dependence on  $\sigma$  of the moments of  $\Theta$ . In fact, for any  $p > 2$

$$\int_{-d}^d |\eta|^p \Theta(\eta) d\eta = E(|\sigma Y|^p) = \sigma^p E(|Y|^p).$$

Note that equation (24) in weak form takes the simpler expression

$$\frac{d}{dt} \int_0^\infty f(w, t) \phi(w) dw = \int_0^\infty \int_{-D}^D \int_{-z}^z \Phi(\mu(S), \xi) \Theta(\eta) f(w, t) (\phi(w') - \phi(w)) d\xi d\eta dw. \quad (29)$$

By a weak solution of the initial value problem for equation (24), corresponding to the initial probability density  $f_0(w) \in \mathcal{M}_p, p > 1$  we shall mean any probability density  $f \in C^1(\mathbb{R}_+, \mathcal{M}_p)$  satisfying the weak form (29) for  $t > 0$  and all  $\phi \in \mathcal{F}_p(\mathbb{R}_+)$ , and such that for all  $\phi \in \mathcal{F}_p(\mathbb{R}_+)$

$$\lim_{t \rightarrow 0} \int_0^\infty f(w, t) \phi(w) dw = \int_0^\infty f_0(w) \phi(w) dw. \quad (30)$$

The form (29) is easier to handle, and it is the starting point to study the evolution of macroscopic quantities (moments). The existence of a weak solution to equation (24) can be seen easily using the same methods available for the linear Boltzmann equation (see [27] and the references therein for example).

From (29) follows the conservation of the total number of investors taking  $\phi(w) = 1$ . The choice  $\phi(w) = w$  is of particular interest since it gives the time evolution of the average wealth which characterizes the price behavior. In fact, the mean wealth is not conserved in the model since we have

$$\frac{d}{dt} \int_0^\infty f(w, t) w dw = \left( r + \mu(S) \left( \frac{S' - S + D}{S} - r \right) \right) \int_0^\infty f(w, t) w dw. \quad (31)$$

Note that since the sign of the right hand side is nonnegative the mean wealth is nondecreasing in time. In particular we can rewrite the equation as

$$\frac{d}{dt} \bar{w}(t) = ((1 - \mu(S))r + \mu(S)\bar{x}(S')) \bar{w}(t). \quad (32)$$

From this we get the equation for the price

$$\frac{d}{dt} S(t) = \frac{\mu(S(t))}{\mu(S(t)) - \dot{\mu}(S(t))S(t)} ((1 - \mu(S(t)))r + \mu(S(t))\bar{x}(S'(t))) S(t), \quad (33)$$

where  $S'$  is given by (20) and

$$\dot{\mu}(S) = \frac{d\mu(S)}{dS} \leq 0.$$

Now since from (22) it follows by the monotonicity of  $\mu$  that

$$\bar{x}(S') \leq M \stackrel{def}{:=} r + \frac{D}{S(0)(1 - \mu(S(0)))},$$

using (32) we have the bound

$$\bar{w}(t) \leq \bar{w}(0) \exp(Mt). \quad (34)$$

From (10) we obtain immediately

$$\frac{S(t)}{\mu(S(t))} \leq \frac{S(0)}{\mu(S(0))} \exp(Mt),$$

which gives

$$S(t) \leq S(0) \exp(Mt). \quad (35)$$

**Remark 4.1** For a constant  $\mu(\cdot) = C$ ,  $C \in (0, 1)$  we have the explicit expression for the grow of the wealth (and consequently of the price)

$$\bar{w}(t) = \bar{w}(0) \exp(rt) - (1 - \exp(rt)) \frac{nD}{1 - C}. \quad (36)$$

Analogous bounds to (34) for moments of higher order can be obtained in a similar way. Let us consider the case of moments of order  $p \geq 2$  which we will need in the sequel. Taking  $\phi(w) = w^p$  we get

$$\frac{d}{dt} \int_0^\infty w^p f(w, t) dw = \int_0^\infty \int_{-d}^d \int_{-z}^z \Phi(\mu(S), \xi) \Theta(\eta) f(w, t) (w'^p - w^p) d\xi d\eta dw. \quad (37)$$

Moreover, we can write

$$w'^p = w^p + pw^{p-1}(w' - w) + \frac{1}{2}p(p-1)\tilde{w}^{p-2}(w' - w)^2,$$

where, for some  $0 \leq \vartheta \leq 1$

$$\tilde{w} = \vartheta w' + (1 - \vartheta)w.$$

Hence,

$$\begin{aligned} & \int_0^\infty \int_{-d}^d \int_{-z}^z \Phi(\mu(S), \xi) \Theta(\eta) f(w, t) (w'^p - w^p) d\xi d\eta dw \\ &= \int_0^\infty \int_{-d}^d \int_{-z}^z \Phi(\mu(S), \xi) \Theta(\eta) f(w, t) (pw^{p-1}(w' - w) + \frac{1}{2}p(p-1)\tilde{w}^{p-2}(w' - w)^2) d\xi d\eta dw \\ &= p((1 - \mu(S))r + \mu(S)\bar{x}(S')) \int_0^\infty w^p f(w, t) dw + \frac{1}{2}p(p-1) \\ & \quad \int_0^\infty \int_{-d}^d \int_{-z}^z \Phi(\mu(S), \xi) \Theta(\eta) f(w, t) \tilde{w}^{p-2} w^2 ((1 - \gamma)r + \gamma x(S', \eta))^2 d\xi d\eta dw. \end{aligned}$$

From

$$\begin{aligned} \tilde{w}^{p-2} &= w^{p-2} (1 + \vartheta((1 - \gamma)r + \gamma x(S', \eta)))^{p-2} \leq w^{p-2} (1 + r + x(S', \eta))^{p-2} \\ &\leq C_p w^{p-2} (1 + r^{p-2} + |x(S', \eta)|^{p-2}) \end{aligned}$$

and

$$((1 - \gamma)r + \gamma x(S', \eta))^2 \leq 2(r^2 + x(S', \eta)^2)$$

we have

$$\begin{aligned} & \int_0^\infty \int_{-d}^d \int_{-z}^z \Phi(\mu(S), \xi) \Theta(\eta) f(w, t) \tilde{w}^{p-2} w^2 d\xi d\eta dw \\ &\leq 2C_p \int_0^\infty \int_{-d}^d \Theta(\eta) f(w, t) w^p (1 + r^{p-2} + |x(S', \eta)|^{p-2}) (r^2 + x(S', \eta)^2) d\eta dw. \end{aligned}$$

Since

$$\int_{-d}^d \Theta(\eta) |x(S', \eta)|^p d\eta \leq \frac{c_p}{S^p} ((S' - S)^p + D^p + \sigma^p E(|Y|^p)) \quad (38)$$

we finally obtain the bound

$$\frac{d}{dt} \int_0^\infty w^p f(w, t) dw \leq A_p(S) \int_0^\infty w^p f(w, t) dw \quad (39)$$

where

$$A_p(S) = 2C_p(r^p + r^{p-2} \left(1 + \frac{c_2}{S^2}((S' - S)^2 + D^2 + \sigma^2 E(|Y|^2))\right) + r^2 \left(1 + \frac{c_{p-2}}{S^{p-2}}((S' - S)^{p-2} + D^{p-2} + \sigma^{p-2} E(|Y|^{p-2}))\right))$$

and  $C_p$ ,  $c_{p-2}$  and  $c_2$  are suitable constants.

We can summarize our results in the following

**Theorem 4.1** *Let the probability density  $f_0 \in \mathcal{M}_p$ , where  $p = 2 + \delta$  for some  $\delta > 0$ . Then, the average wealth is increasing with time exponentially following (34). As a consequence, if  $\mu$  is a non increasing function of  $S$ , the price does not grow more than exponentially as in (35). Similarly, higher order moments do not increase more than exponentially, and we have the bound (39).*

## 5 Fokker-Planck asymptotics and self-similar solution

The previous analysis shows that in general it is difficult to study in details the asymptotic behavior of the system. In addition we must take into account the exponential growth of the average wealth. In this case, one way to get information on the properties of the solution for large time relies in a suitable scaling of the solution. As is usual in kinetic theory, however, particular asymptotics of the equation result in simplified models (generally of Fokker-Planck type), for which it is easier to analyze their behavior. Here following the analysis in [5, 21], inspired by similar asymptotic limits for inelastic gases [8, 24], we consider the limit of large times in which the market originates a very small exchange of wealth (small rates of return  $r$  and  $x$ ).

In order to study the asymptotic behavior of the distribution function  $f(w, t)$  we start from the weak form of the kinetic equation

$$\frac{d}{dt} \int_0^\infty f(w, t) \phi(w) dw = \int_0^\infty \int_{-d}^d \int_{-z}^z \Phi(\mu(S), \xi) \Theta(\eta) f(w, t) (\phi(w') - \phi(w)) d\xi d\eta dw, \quad (40)$$

and consider a second order Taylor expansion of  $\phi$  around  $w$

$$\phi(w') - \phi(w) = w(r + \gamma(x(S', \eta) - r)) \phi'(w) + \frac{1}{2} w^2 (r + \gamma(x(S', \eta) - r))^2 \phi''(\tilde{w}),$$

where, for some  $0 \leq \vartheta \leq 1$

$$\tilde{w} = \vartheta w' + (1 - \vartheta)w.$$

Inserting this expansion in the collision operator, we get

$$\int_0^\infty \int_{-d}^d \int_{-z}^z \Phi(\mu(S), \xi) \Theta(\eta) f(w, t) (\phi(w') - \phi(w)) d\xi d\eta dw$$

$$\begin{aligned}
&= \int_0^\infty \int_{-d}^d \int_{-z}^z \Phi(\mu(S), \xi) \Theta(\eta) f(w, t) w (r + \gamma(x(S', \eta) - r)) \phi'(w) d\xi d\eta dw \\
&+ \frac{1}{2} \int_0^\infty \int_{-d}^d \int_{-z}^z \Phi(\mu(S), \xi) \Theta(\eta) f(w, t) w^2 (r + \gamma(x(S', \eta) - r))^2 \phi''(w) d\xi d\eta dw \\
&+ R_r(S, S')
\end{aligned}$$

where

$$\begin{aligned}
R_r(S, S') &= \frac{1}{2} \int_0^\infty \int_{-d}^d \int_{-z}^z \Phi(\mu(S), \xi) \Theta(\eta) f(w, t) \\
&w^2 (r + \gamma(x(S', \eta) - r))^2 (\phi''(\tilde{w}) - \phi''(w)) d\xi d\eta dw.
\end{aligned} \tag{41}$$

Recalling that  $E[\xi] = 0$ ,  $E[\eta] = 0$ ,  $E[\xi^2] = \zeta^2$  and  $E[\eta^2] = \sigma^2$  we can simplify the above expression to obtain

$$\begin{aligned}
&\int_0^\infty \int_{-d}^d \int_{-z}^z \Phi(\mu(S), \xi) \Theta(\eta) f(w, t) (\phi(w') - \phi(w)) d\xi d\eta dw \\
&= \int_0^\infty f(w, t) w \left( r + \mu(S) \left( \frac{S' - S + D}{S} - r \right) \right) \phi'(w) dw \\
&+ \frac{1}{2} \int_0^\infty f(w, t) w^2 \left( r^2 + (\zeta^2 + \mu(S)^2) \left( \frac{(S' - S)^2}{S^2} + \frac{\sigma^2 + D^2}{S^2} + 2D \frac{S' - S}{S^2} \right. \right. \\
&+ \left. \left. r^2 - 2r \frac{S' - S + D}{S} \right) + 2r\mu(S) \left( \frac{S' - S + D}{S} - r \right) \right) \phi''(w) dw \\
&+ R_r(S, S').
\end{aligned}$$

Now we set

$$\tau = rt, \quad \tilde{f}(w, \tau) = f(w, t), \quad \tilde{S}(\tau) = S(t), \quad \tilde{\mu}(\tilde{S}) = \mu(S)$$

which implies that  $\tilde{f}(w, \tau)$  satisfies the equation

$$\begin{aligned}
&\frac{d}{d\tau} \int_0^\infty \tilde{f}(w, \tau) \phi(w) dw \\
&= \int_0^\infty \tilde{f}(w, \tau) w \left( 1 + \tilde{\mu}(\tilde{S}) \left( \frac{\tilde{S}' + D - \tilde{S}}{r\tilde{S}} - 1 \right) \right) \phi'(w) dw \\
&+ \frac{1}{2} \int_0^\infty \tilde{f}(w, \tau) w^2 \left( r + (\zeta^2 + \tilde{\mu}(\tilde{S})^2) \left( \frac{(\tilde{S}' - \tilde{S})^2}{r\tilde{S}^2} + \frac{\sigma^2 + D^2}{r\tilde{S}^2} + 2D \frac{\tilde{S}' - \tilde{S}}{r\tilde{S}^2} \right. \right. \\
&+ \left. \left. r - 2 \frac{\tilde{S}' + D - \tilde{S}}{\tilde{S}} \right) + 2\tilde{\mu}(\tilde{S}) \left( \frac{\tilde{S}' + D - \tilde{S}}{\tilde{S}} - r \right) \right) \phi''(w) dw \\
&+ \frac{1}{r} R_r(\tilde{S}, \tilde{S}').
\end{aligned}$$

Now we consider the limit of very small values of the constant rate  $r$ . In order for such limit to have a sense and preserve the characteristics of the model we must assume that

$$\lim_{r \rightarrow 0} \frac{\sigma^2}{r} = \nu, \quad \lim_{r \rightarrow 0} \frac{D}{r} = \lambda. \tag{42}$$

First let us note that the above limits in (20) imply immediately that

$$\lim_{r \rightarrow 0} \tilde{S}' = \tilde{S}. \quad (43)$$

We begin to show that the remainder is small for small values of  $r$ .

Since  $\phi \in \mathcal{F}_{2+\delta}(\mathbb{R}_+)$ , and  $|\tilde{w} - w| = \vartheta|w' - w|$

$$|\phi''(\tilde{w}) - \phi''(w)| \leq \|\phi''\|_\delta |\tilde{w} - w|^\delta \leq \|\phi''\|_\delta |w' - w|^\delta. \quad (44)$$

Hence

$$\begin{aligned} \left| \frac{1}{r} R_r(\tilde{S}, \tilde{S}') \right| &\leq \frac{\|\phi''\|_\delta}{2r} \int_0^\infty \int_{-d}^d \int_{-z}^z \Phi(\mu(S(\tau)), \xi) \Theta(\eta) \\ &\quad |(1 - \gamma)r + \gamma x(\tilde{S}', \eta)|^{2+\delta} \tilde{f}(w, \tau) w^{2+\delta} d\xi d\eta dw. \end{aligned}$$

By the inequality

$$|(1 - \gamma)r + \gamma x(\tilde{S}', \eta)|^{2+\delta} \leq 2^{2+\delta} \left( r^{2+\delta} + |x(\tilde{S}', \eta)|^{2+\delta} \right),$$

and (38) we get

$$\begin{aligned} \left| \frac{1}{r} R_r(\tilde{S}, \tilde{S}') \right| &\leq 2^{1+\delta} \|\phi''\|_\delta \cdot \\ &\cdot \left( r^{1+\delta} + \frac{c_{2+\delta}}{r \tilde{S}^{2+\delta}} ((\tilde{S}' - \tilde{S})^{2+\delta} + D^{2+\delta} + \sigma^{2+\delta} E(|Y|^{2+\delta})) \right) \int_0^\infty \tilde{f}(w, \tau) w^{2+\delta} dw. \end{aligned}$$

As a consequence of (42)-(43), from this inequality it follows that  $R_r(\tilde{S}, \tilde{S}')$  converges to zero as  $r \rightarrow 0$ , if

$$\int_0^\infty w^{2+\delta} \tilde{f}(w, \tau) dw$$

remains bounded at any fixed time  $\tau > 0$ , provided the same bound holds at time  $\tau = 0$ . This is guaranteed by inequality (39) since  $A_p(\tilde{S}) \rightarrow 0$  in the asymptotic limit defined by (42).

Next we write

$$\tilde{\mu}(\tilde{S}') = \tilde{\mu}(\tilde{S}) + (\tilde{S}' - \tilde{S}) \dot{\tilde{\mu}}(\tilde{S}) + O((\tilde{S}' - \tilde{S})^2),$$

where

$$\dot{\tilde{\mu}}(\tilde{S}) = \frac{d\tilde{\mu}(\tilde{S})}{d\tilde{S}} \leq 0.$$

Then using the above expansion in (20) from (42) we obtain

$$\lim_{r \rightarrow 0} \frac{\tilde{S}' - \tilde{S}}{r} = \kappa(\tilde{S}) \left( \tilde{S} + \frac{\tilde{\mu}(\tilde{S})}{1 - \tilde{\mu}(\tilde{S})} \lambda \right), \quad (45)$$

with

$$0 < \kappa(\tilde{S}) \stackrel{def}{=} \frac{\tilde{\mu}(\tilde{S})(1 - \tilde{\mu}(\tilde{S}))}{\tilde{\mu}(\tilde{S})(1 - \tilde{\mu}(\tilde{S})) - \tilde{S} \dot{\tilde{\mu}}(\tilde{S})} \leq 1. \quad (46)$$

Now sending  $r \rightarrow 0$  within the same assumptions we obtain the weak form

$$\begin{aligned} & \frac{d}{d\tau} \int_0^\infty \tilde{f}(w, \tau) \phi(w) dw \\ &= \left( 1 + \tilde{\mu}(\tilde{S}) \left( (\kappa(\tilde{S}) - 1) + \frac{\tilde{\mu}(\tilde{S})(\kappa(\tilde{S}) - 1) + 1}{1 - \tilde{\mu}(\tilde{S})} \frac{\lambda}{\tilde{S}} \right) \right) \int_0^\infty \tilde{f}(w, \tau) w \phi'(w) dw \\ & \quad + \frac{1}{2} \frac{(\tilde{\mu}(\tilde{S})^2 + \zeta^2)}{\tilde{S}^2} \nu \int_0^\infty \tilde{f}(w, \tau) w^2 \phi''(w) dw. \end{aligned}$$

This corresponds to the weak form of the Fokker-Planck equation

$$\frac{\partial}{\partial \tau} \tilde{f} + A(\tau) \frac{\partial}{\partial w} (w \tilde{f}) = \frac{1}{2} B(\tau) \frac{\partial^2}{\partial w^2} (w^2 \tilde{f}),$$

or equivalently

$$\frac{\partial}{\partial \tau} \tilde{f} = \frac{\partial}{\partial w} \left[ -A(\tau) w \tilde{f} + \frac{1}{2} B(\tau) \frac{\partial}{\partial w} w^2 \tilde{f} \right], \quad (47)$$

with

$$A(\tau) = 1 + \tilde{\mu}(\tilde{S}) \left( (\kappa(\tilde{S}) - 1) + \frac{\tilde{\mu}(\tilde{S})(\kappa(\tilde{S}) - 1) + 1}{1 - \tilde{\mu}(\tilde{S})} \frac{\lambda}{\tilde{S}} \right) \quad (48)$$

$$B(\tau) = \frac{(\tilde{\mu}(\tilde{S})^2 + \zeta^2)}{\tilde{S}^2} \nu. \quad (49)$$

Thus we have proved

**Theorem 5.1** *Let the probability density  $f_0 \in \mathcal{M}_p$ , where  $p = 2 + \delta$  for some  $\delta > 0$ . Then, as  $r \rightarrow 0$ ,  $\sigma \rightarrow 0$ ,  $D \rightarrow 0$  in such a way that  $\sigma^2 = \nu r$  and  $D = \lambda r$  the weak solution to the Boltzmann equation (29) for the scaled density  $\tilde{f}_r(w, \tau) = f(v, t)$ , with  $\tau = rt$  converges, up to extraction of a subsequence, to a probability density  $\tilde{f}(w, \tau)$ . This density is a weak solution of the Fokker-Planck equation (47).*

We remark that even for the Fokker-Planck model the mean wealth is increasing with time. A simple computation show that

$$\dot{w}(\tau) = \frac{d}{d\tau} \int_0^\infty \tilde{f}(w, \tau) w dw = A(\tau) \int_0^\infty \tilde{f}(w, \tau) w dw = A(\tau) \bar{w}(\tau). \quad (50)$$

Using (46) we get the bound

$$(1 - \tilde{\mu}(\tilde{S})) \bar{w}(\tau) + n\lambda \leq \dot{w}(\tau) \leq \bar{w}(\tau) + \frac{n\lambda}{1 - \tilde{\mu}(\tilde{S})}. \quad (51)$$

Similarly for the second order moment we have

$$\dot{e}(\tau) = \frac{d}{d\tau} \int_0^\infty \tilde{f}(w, \tau) w^2 dw = (2A(\tau) + B(\tau)) \int_0^\infty \tilde{f}(w, \tau) w^2 dw = (2A(\tau) + B(\tau)) \bar{e}(\tau). \quad (52)$$

In order to search for self-similar solutions we consider the scaling

$$\tilde{f}(w, \tau) = \frac{1}{w} \tilde{g}(\chi, \tau), \quad \chi = \log(w).$$

Simple computations show that  $\tilde{g}(\chi, \tau)$  satisfy the linear convection-diffusion equation

$$\frac{\partial}{\partial \tau} \tilde{g}(\chi, \tau) = \left( \frac{B(\tau)}{2} - A(\tau) \right) \frac{\partial}{\partial \chi} \tilde{g}(\chi, \tau) + \frac{B(\tau)}{2} \frac{\partial^2}{\partial \chi^2} \tilde{g}(\chi, \tau),$$

which admits the self-similar solution (see [16] for example)

$$\tilde{g}(\chi, \tau) = \frac{1}{(2b(\tau)\pi)^{1/2}} \exp \left( -\frac{(\chi + b(\tau)/2 - a(\tau))^2}{2b(\tau)} \right), \quad (53)$$

where

$$a(\tau) = \int_0^\tau A(s) ds + C_1, \quad b(\tau) = \int_0^\tau B(s) ds + C_2.$$

Reverting to the original variables we obtain the lognormal asymptotic behavior of the model

$$\tilde{f}(w, \tau) = \frac{1}{w(2b(\tau)\pi)^{1/2}} \exp \left( -\frac{(\log(w) + b(\tau)/2 - a(\tau))^2}{2b(\tau)} \right). \quad (54)$$

The constants  $C_1 = a(0)$  and  $C_2 = b(0)$  can be determined from the initial data at  $t = 0$ . If we denote with  $\bar{w}(0)$  and  $\bar{e}(0)$  the initial value of the first two central moments we get

$$C_1 = \log(\bar{w}(0)), \quad C_2 = \log \left( \frac{\bar{e}(0)}{(\bar{w}(0))^2} \right).$$

Finally a direct computation shows that

$$a(\tau) = \int_0^\tau \frac{\dot{\bar{w}}(s)}{\bar{w}(s)} ds + C_1 = \log(\bar{w}(\tau)), \quad (55)$$

and

$$b(\tau) = \int_0^\tau \left( \frac{\dot{\bar{e}}(s)}{\bar{e}(s)} - 2 \frac{\dot{\bar{w}}(s)}{\bar{w}(s)} \right) ds + C_2 = \log \left( \frac{\bar{e}(\tau)}{(\bar{w}(\tau))^2} \right). \quad (56)$$

### Remark 5.1

- If we assume  $\zeta$  and  $\sigma$  of the same order of magnitude in the Fokker-Planck limit the noise introduced by the agents deviations with respect to their optimal behavior does not play any role and the only source of diffusion is due to the stochastic nature of the returns.
- In the case of constant investments  $\tilde{\mu}(\cdot) = C$ ,  $C \in (0, 1)$  we have the simplified Fokker-Planck equation

$$\frac{\partial}{\partial \tau} \tilde{f} = \frac{\partial}{\partial w} \left[ - \left( 1 + \frac{C}{1-C} \frac{\lambda}{\tilde{S}} \right) w \tilde{f} + \frac{1}{2} \frac{(C^2 + \zeta^2)}{\tilde{S}^2} \nu \frac{\partial}{\partial w} w^2 \tilde{f} \right].$$

It is easy to verify that for such simplified situation the couple of ordinary differential equations for the evolution of the first two central moments (50) and (52) can be solved explicitly.



- Imposing the conservation of the mean wealth with the scaling

$$\tilde{f}(w, \tau) = \frac{\bar{w}(0)}{\bar{w}(\tau)} \hat{f}(v, \tau), \quad v = \frac{\bar{w}(0)}{\bar{w}(\tau)} w \quad (57)$$

we have the diffusion equation

$$\frac{\partial}{\partial \tau} \hat{f}(v, \tau) = \frac{B(\tau)}{2} \frac{\partial^2}{\partial v^2} (v^2 \hat{f}(v, \tau)).$$

This yields the asymptotic lognormal behavior

$$\hat{f}(v, \tau) = \frac{1}{v(2 \log(\bar{E}(\tau)/\bar{w}(0)^2)\pi)^{1/2}} \exp \left( -\frac{(\log(v) + \log(\sqrt{\bar{E}(\tau)}/\bar{w}(0)))^2}{2 \log(\bar{E}(\tau)/\bar{w}(0)^2)} \right), \quad (58)$$

with

$$\int_0^\infty \hat{f}(v, \tau) v dv = \bar{w}(0), \quad \bar{E}(\tau) = \int_0^\infty \hat{f}(v, \tau) v^2 dv.$$

## 6 Numerical examples

In this section we report the results of different numerical simulations of the proposed kinetic equations. In all the numerical tests we use  $N = 1000$  agents and  $n = 10000$  shares. Initially each investor has a total wealth of 1000 composed of 10 shares, at a value of 50 per share, and 500 in bonds. The random variables  $\xi$  and  $\eta$  have been supposed distributed accordingly to truncated normal distributions so that negative wealth values are avoided (no borrowing and no short selling). In Test 1 and 2 we compare the results obtained with the Monte Carlo simulation of the kinetic model together with a direct solution of the price equation (33). In the last test case we consider the time averaged Monte Carlo asymptotic behavior of the kinetic model and compare its numerical self-similar solution with the explicit one computed in the last section using the Fokker-Planck model.

### Test 1

In the first test we take a riskless interest rate  $r = 0.01$  and an average dividend growth rate  $D = 0.015$  and assume that the agents simply follow a constant investments rule,  $\mu(\cdot) = C$ , with  $C \in (0, 1)$  constant. As a consequence of our choice of parameters we have  $C = 0.5$  and the evolution of the mean wealth and of the price in the kinetic model are known explicitly (36). We report the results after 400 stock market iterations with  $\xi$  and  $\eta/S(0)$  distributed with standard deviation 0.2 and 0.3 respectively. In Figure 2 we report the simulated price behavior together with the evolution computed from (33). The fraction of investments in time in the Monte Carlo simulation fluctuates around its optimal value and is given in Figure 3.

### Test 2

In the next test case we take the same parameters as in Test 1 but with a non constant profile  $\mu(\cdot)$ . More precisely we take a monotone decreasing exponential law

$$\mu(S) = C_1 + (1 - C_1)e^{-C_2 S}$$

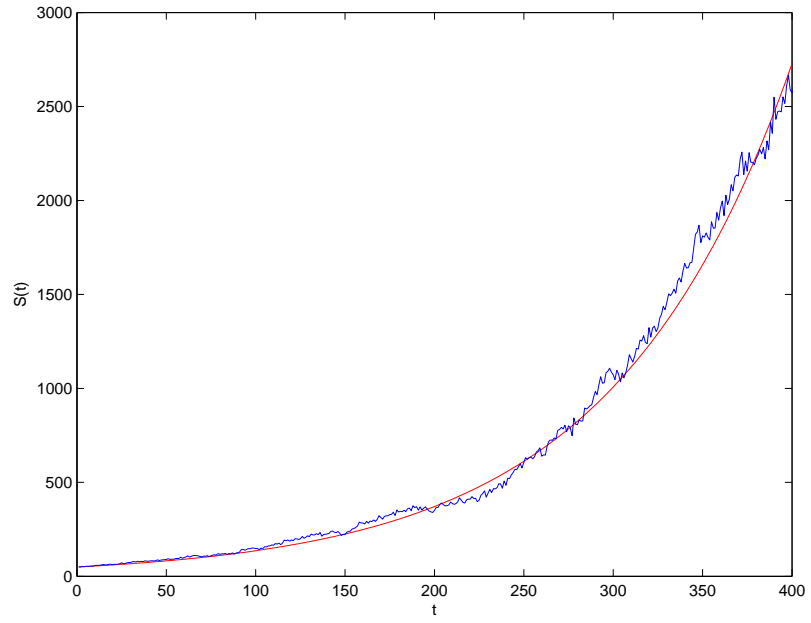


Figure 2: Test 1. Exponential grow of the price in time. Numerical simulation of the kinetic model.

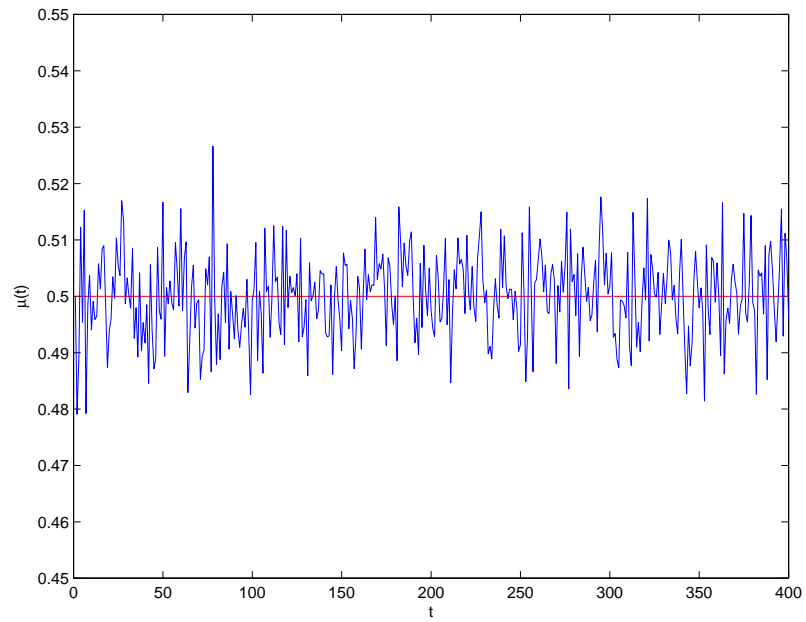


Figure 3: Test 1. Fluctuations of the corresponding fraction of investments in time.

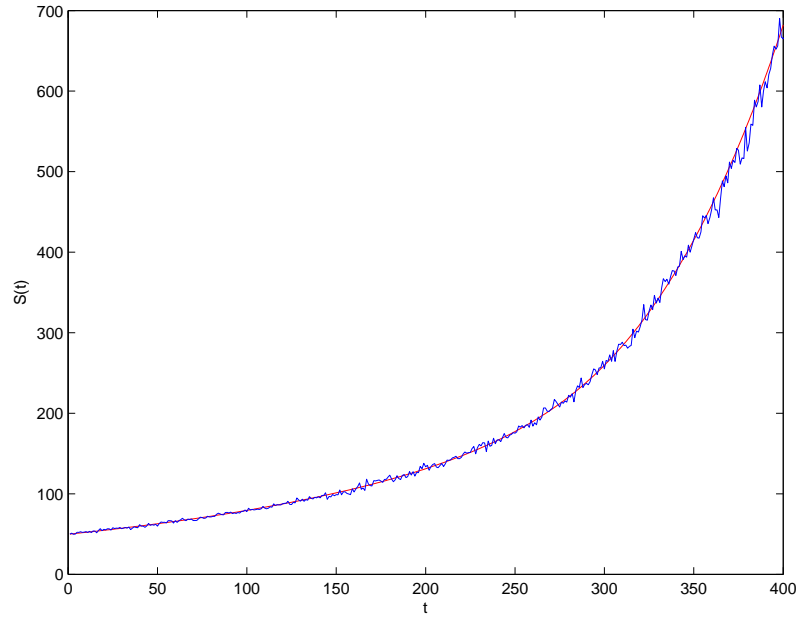


Figure 4: Test 2. Price behavior in time. Numerical simulation of the the kinetic model.

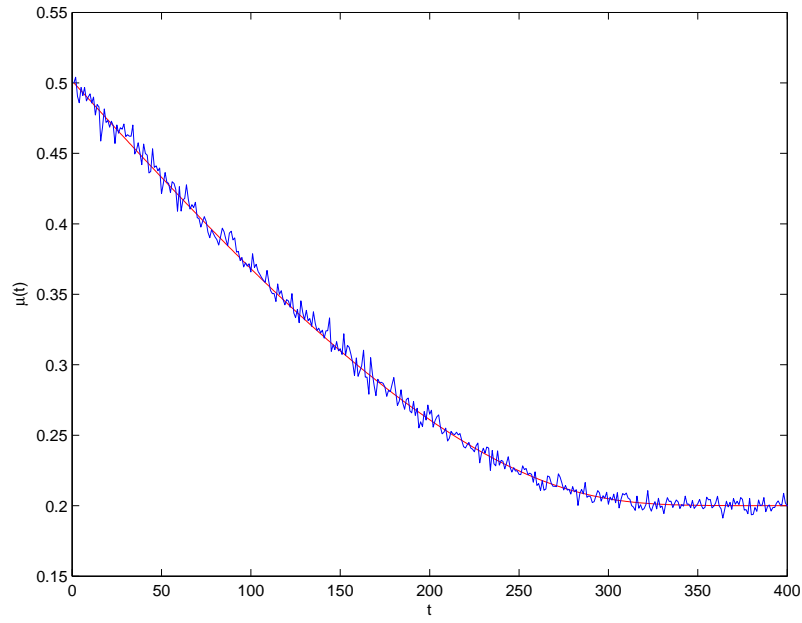


Figure 5: Test 2. Fluctuations of the corresponding fraction of investments in time.

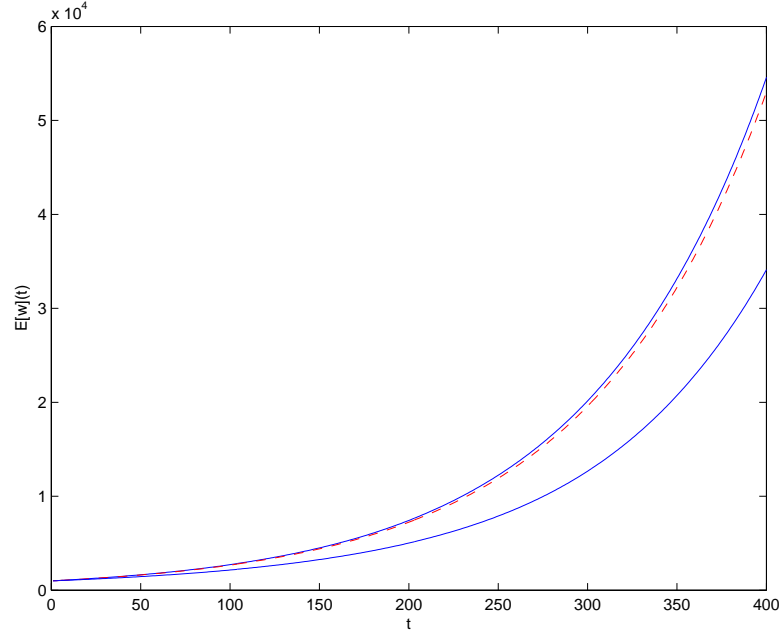


Figure 6: Test 1 and 2. Behavior of the mean wealth in the kinetic model. Top curve refers to Test 1, bottom curve to Test2. The dashed line corresponds to the exponential growth at a rate equal to  $r$ .

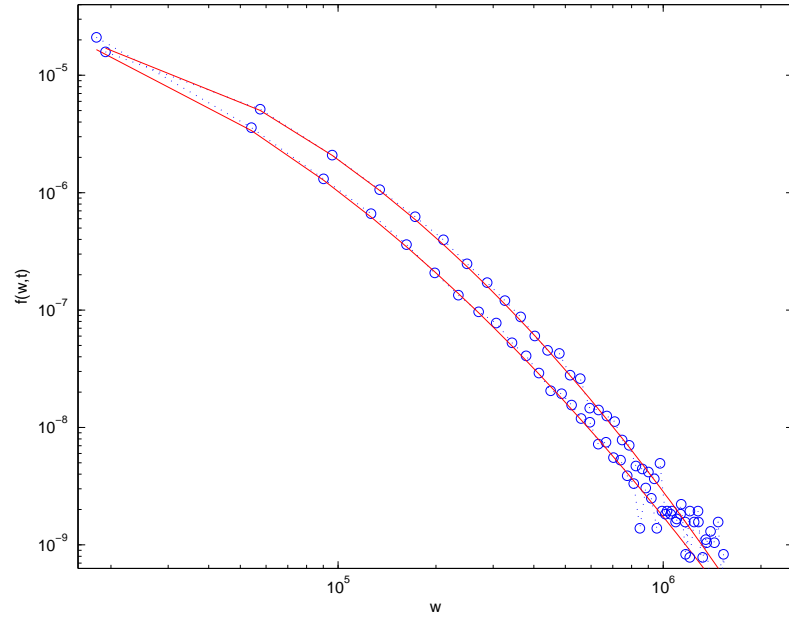


Figure 7: Test 1 and 2. Log-log plot of the mean wealth distribution together with a lognormal fitting. Top curves refers to Test 1, bottom curves to Test2.

with  $C_1 = 0.2$  and  $C_2 = \log((1 - C_1)/(0.5 - C_1))/S_0 \approx 0.02$  so that the price equation is satisfied for  $S_0 = 50$ . We have  $0.2 < \mu(\cdot) \leq 0.5$ . The results for the price evolution and the investments behavior are plotted in Figures 4 and 5. The solution for the price in the kinetic equation has been computed by direct numerical discretization of (33). Note that the final price is approximatively 5 times smaller then the one in the constant investment case with  $\mu = 0.5$ . In Figure 6 we compare the behavior of the mean wealth in Test 1 and 2 together with the exponential grow at a rate  $r$  obtained with simple investments in bonds. We can observe that the time decay of investments in Test 2 is fast enough to produce a wealth grow below the rate  $r$ . On the contrary, as observed in Section 3, a constant investment strategy produces a curve above such rate. Finally in Figure 7 we plot the averaged wealth distributions at the final computation time in log-log scale together with a lognormal fitting. The results show a lognormal behavior of the tails even for the Boltzmann model. Note that thanks to equation (3) the same distribution is observed for the number of stocks owned by the agents.

### Test 3

In the last test case we consider the asymptotic limit of the Boltzmann model and compare its numerical self-similar solution with the explicit one computed in the last section using the Fokker-Planck model. To this goal we consider the self similar scaling (57) and compute the solution for the values  $r = 0.001$ ,  $D = 0.0015$  with  $\xi$  and  $\eta/S(0)$  distributed with standard deviation 0.05. We report the numerical solution for a constant value of  $\mu = 0.5$  at different times  $t = 50, 200, 500$  in Figures 8 and 9. A very good agreement between the Boltzmann and the lognormal Fokker-Planck solutions is observed as expected from the results of the last section. We also compute the corresponding Lorentz curve  $L(F(w, t))$  defined as

$$L(F(w, t)) = \frac{\int_0^w f(v, t) v dv}{\int_0^\infty f(v, t) v dv}, \quad F(w, t) = \int_0^w f(v, t) dv,$$

and the Gini coefficient  $G \in [0, 1]$

$$G = 1 - 2 \int_0^1 L(F(w, t)) dw.$$

The Gini coefficient is a measure of the inequality in the wealth distribution [10]. A value of 0 corresponds to the line of perfect equality depicted in Figure 10 together with the different Lorentz curves. It is clear that inequalities grow in time due to the speculative dynamics.

## 7 Conclusions

We have derived a simple linear mesoscopic model which describes a financial market under the assumption that the distribution of investments is known as a function of the price. The model is able to describe the exponential grow of the price of the stock and the grow of the wealth above the rate produced by simple investments in bonds. The long time

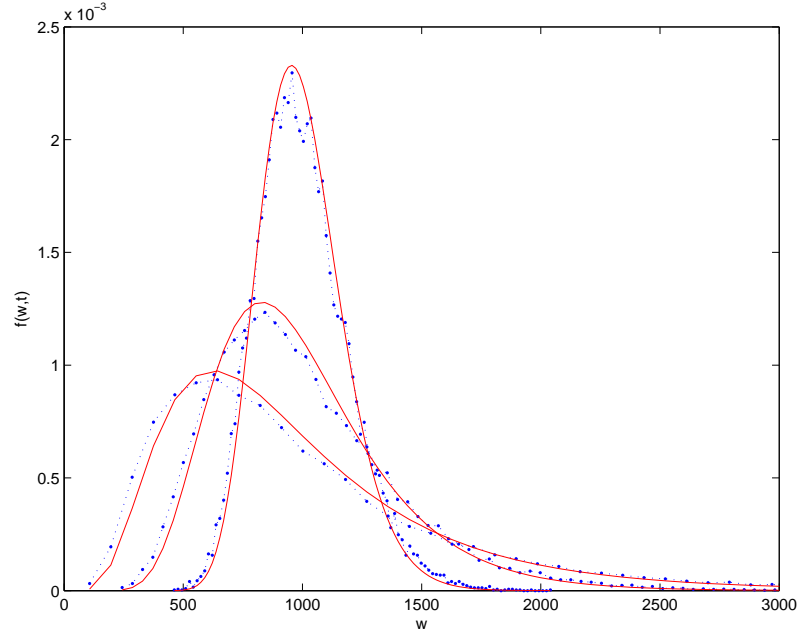


Figure 8: Test 3. Distribution function at  $t = 50, 200, 500$ . The continuous line is the lognormal Fokker-Planck solution.

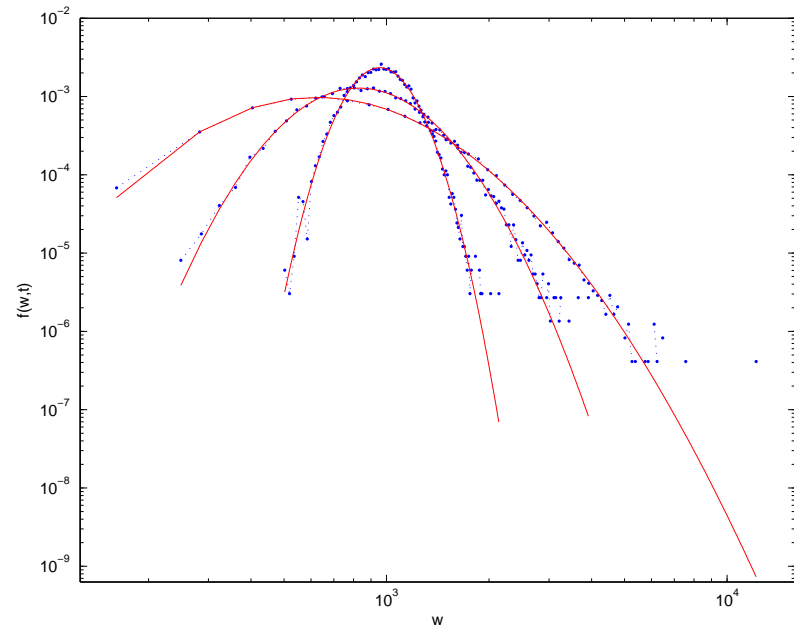


Figure 9: Test 3. Log-log plot of the distribution function at  $t = 50, 200, 500$ . The continuous line is the lognormal Fokker-Planck solution.

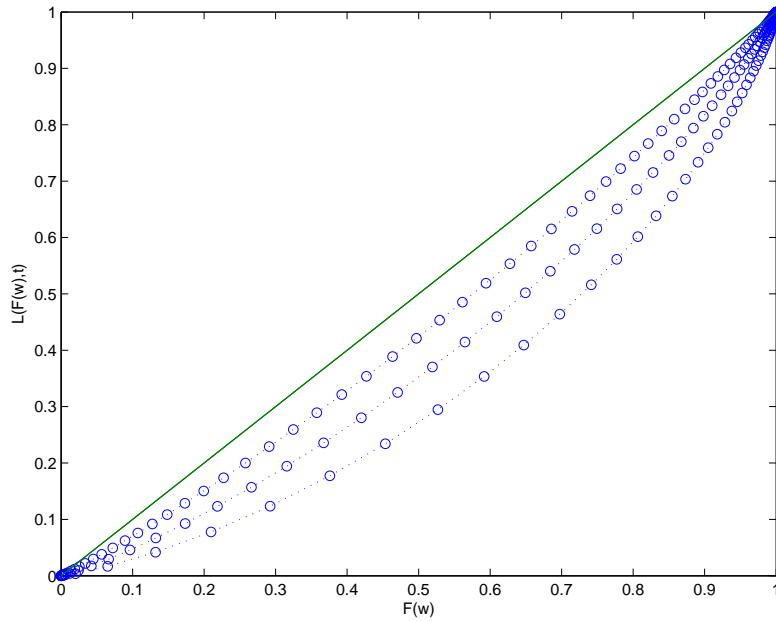


Figure 10: Test 3. The corresponding Lorenz curves. The Gini coefficients are  $G = 0.1$ ,  $G = 0.2$  and  $G = 0.3$  respectively.

behavior of the model has been studied with the help of a Fokker-Planck approximation. The emergence of a power law tail for the wealth distribution of lognormal type has been proved. In order to produce the effect of a real financial market like booms, cycles and crashes, the distribution of investments should be a function of time (the decision making should be done by maximizing the expected utility) and one should consider heterogeneous populations of investors as in [13, 14]. In this case the model should be modified and consider the time evolution of  $\mu(S, t)$ . Another interesting research direction is related to the possibility to introduce into the model stock options and to relate the kinetic approach to Black-Scholes type equations. All these subjects are actually under investigation and we hope to present other challenging results in the nearby future.

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